

Symmetry, singularities and integrability in complex dynamics VII: Integrability Properties of FRW-Scalar Cosmologies

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February 7, 2008

Abstract

This paper considers some physically interesting cosmological dynamical systems in the FRW-scalarfield category which are examined for integrability according to the criterion of Painlevé. In the literature these systems have been examined from the point of view of dynamical systems and the results from the two disparate methods of analysis are compared. This allows some more general comments to be made on the use of the Painlevé method in covariant systems.

1 Introduction

This is the seventh in a series of papers [1, 2, 3, 4, 5, 6] devoted to various aspects of the three mathematically disparate topics of symmetry, singularities and integrability of dynamical systems. It is the purpose of the series to cement the concepts into a greater obvious unity whereby they are recognised as different aspects of the central properties of dynamical systems which distinguish integrable systems from nonintegrable systems.

The plan of the present paper is as follows. In §2 we present a brief resumé of the analysis of Painlevé with particular reference to some of the subtleties required to deal with systems which do not fit into the neat scheme of the

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standard ARS algorithm [7, 8, 9, 10]. In §§3-6 we present the analysis of a number of models chosen from some interesting areas of Cosmology which have been analysed from the viewpoint of Dynamical Systems. In the approach used in Dynamical Systems the original system of equations is replaced by an n -dimensional system of first order equations, maybe subject to a constraint to preserve the actual number of degrees of freedom. This is not always a suitable format for the application of the Painlevé analysis and we find it necessary to rearrange the systems into a more suitable format. The possession of the Painlevé Property, i.e., the existence of a solution analytic except at moveable polelike singularities (branch point singularities in the case of the ‘weak’ property), is representation dependent, being preserved under a homeographic transformation only, and consequently we must be careful in comparing the results obtained using the Painlevé approach with those obtained using the methods of Dynamical Systems. Finally, in §7, we present some general observations of the results of our analysis paying particular attention to the uses of Painlevé methods and analyses when dealing with general relativistic systems.

2 Methodology of singularity analysis

For a detailed account of our approach to the Painlevé analysis the reader is referred to our earlier paper [4]. Here we present a summary of that account so that the paper is self-contained for the reader not concerned with the fine detail of the ideas behind the singularity analysis.

The basis of the singularity analysis of a system of differential equations is found in the dominant behaviour of the system at a singularity and also the next to leading order behaviour in the neighbourhood of the singularity [11, 12]. When the analysis is concerned with the integrability of a system, the conditions on the behaviour must be tightened to analytic behaviour in the complex plane or at least substantial portions of it. (One can treat with branch point singularities provided the branch cuts do not become too concentrated in the region of the complex plane relevant to the solution of the problem at hand.) This was the original approach by the originators of this field – Kowalevskaya [13] and Painlevé [14, 15, 16]. The idea behind the analysis is that a system of differential equations is integrable if there exists a set of analytic functions which satisfy the system of differential equations. Since an analytic function can be expressed in terms of Laurent series, the demonstration that there exist Laurent expansions about movable singularities containing the requisite number of arbitrary constants for the dependent variables in the system of differential equations is taken to indicate that the system is integrable. This is the basis of the ARS algorithm developed by Ablowitz, Ramani and Segur [7, 8, 9] and presented, for example, by Ramani *et al* [10] and Tabor [17].

The essence of the ARS algorithm is that the solution of an n -dimensional system of equations

$$\dot{x}_i = f_i(t, x), \quad (2.1)$$

where the functions f_i are rational in the dependent variables and algebraic in the independent variable, can be written as either

$$x_i(t) = \sum_{j=0}^{\infty} a_j \tau^{-p_i+jq} \quad (2.2)$$

in the case of a Right Painlevé Series or

$$x_i(t) = \sum_{j=0}^{\infty} a_j \tau^{-p_i-jq} \quad (2.3)$$

in the case of a Left Painlevé Series [18, 11], where $\tau = t - t_0$ and t_0 is the location of the movable singularity, the exponents p_i are strictly positive integers (rational numbers in the case of the so-called weak Painlevé test), the parameter q is a positive integer (respectively rational number) and in the coefficients there are $n - 1$ arbitrary constants which, together with t_0 , give the required number of arbitrary constants for the general solution of the system (2.1). In the case of the Left Painlevé Series, which has the nature of an asymptotic expansion in the region of t infinite (2.3) is usually written as

$$x_i(t) = \sum_{j=0}^{\infty} a_j t^{-p_i-jq} \quad (2.4)$$

and the coefficients now contain n arbitrary constants.

The ARS algorithm provides a mechanistic approach to the establishment of the existence of (2.2) and/or (2.4), which is not always satisfactory [19], but, when handled with sensitivity rather than automatically, provides a useful tool in the analysis of differential equations. The first step is to determine the leading order behaviour by means of the *Ansatz*

$$x_i = \alpha_i \tau^{-p_i}, \quad (2.5)$$

which is substituted into the system (2.1), and to assemble all possible patterns of singular behaviour. For each of these possible patterns the coefficients α_i are calculated. For each pattern the substitution

$$x_i = \alpha_i \tau^{-p_i} + \mu_i \tau^{r-p_i} \quad (2.6)$$

is made to determine the ‘resonances’ r at which the arbitrary constants, μ_i , are introduced. These are determined from the leading order terms of (2.1) by a linearisation process.

For each pattern of singularity behaviour and its concomitant resonances and arbitrary constants the substitution

$$x_i = \sum_{j=0}^{j=j_{max}} a_j \tau^{-p_i+jq}, \quad (2.7)$$

where $j_{max}q = r_{max}$ and r_{max} is the largest resonance, is made into the full system (2.1) to ensure that there is consistency among the resonances and with the nondominant terms of the original system. If this be the case, that pattern of singular behaviour passes the Painlevé test and, if this be the case for all possible patterns of singularity behaviour, the system (2.1) possesses the Painlevé Property which guarantees integrability in most cases. There are exceptions to this general statement and the interested reader is referred to Miritzis *et al* [4] and the references cited therein for a more detailed discussion of the finer points of the application of the ARS algorithm to the techniques of singularity analysis.

There are systems which cannot possess the Painlevé Property since they do not have movable singularities in every variable. We speak not of those for which one or more of the variables has a leading order behaviour in terms of a positive rational exponent. This becomes singular after a sufficient number of differentiations. Rather we refer to those variables which are constant at the movable singularity, t_0 , of other variables. (Positive integral behaviour can be rectified by the simple, yet homeomorphic, means of taking the inverse as a new variable.) Usually an analysis of these cases by means of the *Ansatz* of a series expansion – the ARS algorithm is not applicable in such a case and, indeed, is downright misleading – produces series with fewer than the required number of arbitrary constants. Naturally such series cannot represent the general solution and give no indication of the integrability of the system. However, subject to the convergence of the series, they can indicate some sort of partial solution. For reasons explained in Miritzis *et al* [4] we refer to these solutions as ‘peculiar’ solutions – a literal translation from the Greek usage – to avoid some of the associations with other terms commonly used such as ‘singular’ and ‘particular’. A peculiar solution is one consisting of series which satisfy the system (2.1) containing fewer than the required number of arbitrary constants. Finally we note that the considerations presented here in terms of the systems of first order equations, (2.1), apply *mutatis mutandis* to all systems of ordinary differential equations.

3 Massive scalar field

Consider the Friedmann equations for the evolution of the scale factor $a(t)$ of an open, flat or closed ($k = -1, 0, +1$ respectively) universe in the case where matter is represented by a self-interacting scalar field with potential $V(\phi)$. They are

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} &= \frac{1}{3} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right) \\ 2\frac{\ddot{a}}{a} &= -\frac{1}{3} \left(2\dot{\phi}^2 - V(\phi) \right) \\ \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) &= 0 \end{aligned} \tag{3.1}$$

In this Section, we focus in the particular case that of the scalar potential, $V(\phi)$, is quadratic of the form $\frac{1}{2}m^2\phi^2$ so that the above equations become

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} &= \frac{1}{3} \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 \right) \\ \frac{\ddot{a}}{a} &= -\frac{1}{6} \left(2\dot{\phi}^2 - \frac{1}{2}m^2\phi^2 \right) \\ \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + m^2\phi &= 0.\end{aligned}\tag{3.2}$$

These three equations are not independent since, for example, the first and third of (3.2) lead to the second. (This is also the case in (3.1). The result is independent of the precise form of the scalar field.) We make the choice (3.2c) and so treat the system

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} &= \frac{1}{6} \left(\dot{\phi}^2 + m^2\phi^2 \right) \\ \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + m^2\phi &= 0.\end{aligned}\tag{3.3}$$

We can make some ‘cosmetic’ improvements to (3.3) by rescaling time, t , scalar field, ϕ and scale factor (in the case $k \neq 0$), a , according to

$$\bar{t} = mt, \quad v^2 = \frac{1}{6}\phi^2 \quad \text{and} \quad u = \frac{ma^2}{k}\tag{3.4}$$

to obtain the system

$$\begin{aligned}\dot{u}^2 &= u^2 (\dot{v}^2 + v^2) \\ u\ddot{v} + 3\dot{u}\dot{v} + uv &= 0\end{aligned}\tag{3.5}$$

in the case $k = 0$ and the system

$$\begin{aligned}\dot{u}^2 + 4u &= 4u^2 (\dot{v}^2 + v^2) \\ 2u\ddot{v} + 3\dot{u}\dot{v} + 2uv &= 0\end{aligned}\tag{3.6}$$

in the case that $k \neq 0$. In both instances the overdot now represents differentiation with respect to the new time variable, \bar{t} . We observe that the sign of k is immaterial in the transformation (3.4).

We commence the testing of (3.5) for the possession of the Painlevé Property with the standard substitution

$$a = \alpha\tau^p \quad v = \beta\tau^q,\tag{3.7}$$

where $\tau = \bar{t} - \bar{t}_0$ and \bar{t}_0 represents the location of the moveable singularity if p is found to be negative, to determine the leading order behaviour. We obtain the following exponents for the several terms in (3.5a) and (3.5b) respectively

$$\begin{array}{ccc} 2p-2 & 2p+2q-2 & 2p+2q \\ p+q-2 & p+q-2 & p+q. \end{array}\tag{3.8}$$

From (3.8a) we see that p is arbitrary and that balancing between the first and third terms is possible for $q = -1$. However, then the second term has the lower exponent $2p - 4$ and cannot balance if one is thinking in terms of a Right Painlevé Series. However, there is no problem if one thinks in terms of the existence of a Left Painlevé Series since now the second term vanishes more rapidly than the two dominant terms at infinity as required. Unfortunately the third term of (3.8b) is incompatible with a Left Painlevé Series and one must conclude that $q = 0$ which thereby ends the validity of the next to leading order analysis. The standard procedure outlined in the ARS algorithm ceases to be applicable.

For (3.6) we apply the same methodology as for (3.5) and obtain the same dismal results. There is no possibility of consistent singular behaviour.

In the cases of both (3.5) and (3.6) one can look for the existence of a standard series solution in the absence of the Laurent series implicit in the Painlevé analysis. Indeed we find

$$\begin{aligned} a &= a_0 + a_1 \bar{t} + \frac{3a_0^2 b_0^2 - 2a_1^2}{2a_0} \bar{t}^2 + \dots \\ v &= b_0 + \frac{\sqrt{a_1^2 - a_0^2 b_0^2}}{a_0} \bar{t} - \frac{1}{2} \left(b_0 + 3 \frac{a_1}{a_0} b_1 \right) \bar{t}^2 + \dots \end{aligned} \quad (3.9)$$

in the case of (3.5) and

$$\begin{aligned} u &= a_0 + a_1 \bar{t} + \left(3a_0 b_0^2 - 2 - \frac{a_1^2}{4a_0} \right) \bar{t}^2 + \dots \\ v &= b_0 + \frac{\sqrt{4a_0 + a_1^2 - 4a_0^2 b_0^2}}{2a_0} \bar{t} - \frac{1}{4} \left(2b_0 + 3 \frac{a_1}{a_0} b_1 \right) \bar{t}^2 + \dots \end{aligned} \quad (3.10)$$

in the case of (3.6). The coefficients a_0 , b_0 and a_1 are arbitrary in both cases. Coefficients of higher powers of \bar{t} become somewhat more complicated expressions.

We conclude that the model of the massive scalar field with a quadratic potential described by (3.3) is not integrable in the sense of Painlevé no matter the curvature (the change of variables in (3.4) can at most have the effect of a change from possession of the Painlevé Property to possession of the ‘weak’ Painlevé Property). This is of course not surprising given that very little is now about the evolution of this system in general.

However, making the usual slow-roll approximation, which basically amounts to neglecting the $\ddot{\phi}$ and kinetic energy terms from the Friedman equations, that is

$$\dot{\phi} \sim \frac{V'}{\sqrt{V}} \quad (3.11)$$

implies that in the quadratic potential case we are considering, the exponents may balance with p arbitrary and $q = -1$. This again does not obviously produce some great difference in the possession of the Painlevé property but

it does show that at least the ARS algorithm can be applied. So we see that precisely this condition, which is conducive to inflationary solutions, makes some of the difficulties disappear.

4 Models with an exponential potential

4.1 The scalar-vacuum case

The equations for the simplest case, that of an FRW model with a single scalar field with an exponential potential $V = e^{-\lambda\phi}$ and no other matter source, are [20]:

$$\begin{aligned}\ddot{a} &= -\dot{a}^2 - 2\dot{\phi}^2 + \frac{1}{2}\lambda\dot{a}\dot{\phi} + 1 \\ \ddot{\phi} &= \frac{1}{2}\lambda\dot{\phi}^2 - 3\dot{a}\dot{\phi} + \frac{1}{2}\lambda\end{aligned}\tag{4.1}$$

subject to the constraint

$$\dot{a}^2 = \dot{\phi}^2 + 1 - ke^{-2a+\lambda\phi},\tag{4.2}$$

where, as usual, $k = 0, \pm 1$ depending upon whether the scalar curvature is zero, positive or negative. As we noted in §3 in the instance of equations (3.1), (4.1) and (4.2) are not independent since the differentiation of (4.2) and one of (4.1) leads to the other. However, (4.1) and one of (4.1) and (4.2) are not equivalent systems since one is of the fourth order and the other of the third order.

4.1.1 The scalar-vacuum case as a third order system

For our third order system we take (4.1b) and (4.2), i.e., we consider the system

$$\begin{aligned}\ddot{\phi} &= \frac{1}{2}\lambda\dot{\phi}^2 - 3\dot{a}\dot{\phi} + \frac{1}{2}\lambda \\ \dot{a}^2 &= \dot{\phi}^2 + 1 - ke^{-2a+\lambda\phi}.\end{aligned}\tag{4.3}$$

We transform the system (4.3) into one suitable for application of the singularity analysis by means of the nonhomeographic transformation

$$u = e^{-2a+\lambda\phi} \quad v = e^{\lambda\phi}\tag{4.4}$$

to obtain

$$2u\ddot{v} - 3\dot{u}\dot{v} - \lambda^2 uv = 0\tag{4.5}$$

$$\dot{u}^2 v^2 - 2u\dot{u}\dot{v} + \left(1 - \frac{4}{\lambda^2}\right) u^2 \dot{v}^2 = 4u^2 v^2 - 4ku^3 v^2\tag{4.6}$$

The dominant expressions in the leading order analysis are

$$2\alpha\beta q(q-1)\tau^{p+q-2} - 3\alpha\beta pq\tau^{p+q-2}\tag{4.7}$$

and

$$\alpha^2 \beta^2 p^2 \tau^{2p+2q-2} - 2\alpha^2 \beta^2 pq \tau^{2p+2q-2} + \left(1 - \frac{4}{\lambda^2}\right) \alpha^2 \beta^2 q^2 \tau^{2p+2q-2} + 4k\alpha^3 \beta^2 p^2 \tau^{3p+2q} \quad (4.8)$$

for (4.5a) and (4.5b) respectively after the usual substitutions are made. From (4.8) we obtain $p = -2, q$ arbitrary and, using this in (4.7), we find that $q = -2, 0$.

Firstly we consider the pattern of singular behaviour $(-2, -2)$. From (4.8) we obtain $\alpha = 4/k\lambda^2$ and β remains arbitrary. When we make the substitution

$$u = \alpha \tau^{-2} + \mu \tau^{r-2} \quad v = \beta \tau^{-2} + \nu \tau^{r-2}, \quad (4.9)$$

the condition that the required constants are arbitrary is that

$$\begin{vmatrix} 6r & 2r(r-2) \\ -\frac{16}{\lambda^2} & \frac{16r}{\lambda^2} \end{vmatrix} = 0 \quad (4.10)$$

which has the solutions $r = -1, 0$. The first is generic, the second reflects the arbitrariness of β and there is no third so that the system cannot pass the Painlevé test with this pattern of singularity behaviour. The solution is peculiar (*cf* [4]) with the first few terms being given by

$$\begin{aligned} u &= \frac{4}{k\lambda^2\tau^2} \left(1 + \frac{(\lambda\tau)^2}{12} + \frac{(\lambda\tau)^4}{240} + \dots \right) \\ v &= \frac{b_0}{\tau^2} \left(1 + \frac{(\lambda\tau)^2}{12} + \frac{(\lambda\tau)^4}{240} + \dots \right) \end{aligned} \quad (4.11)$$

which is suggestive of the solution of (4.5) when $v = \text{const}u$ and the pole is fixed at $t = 0$.

The singularity pattern $(-2, 0)$ is without the ambit of the Painlevé test. If we make the *Ansatz*

$$u = \sum_{i=0} a_i \tau^{i-2} \quad v = \sum_{i=0} b_i \tau^i, \quad (4.12)$$

the requirement that the coefficients of the first two terms balance gives

$$a_0 b_1 = 0 \quad a_0^2 b_0^2 + k a_0^3 b_0^2 = 0. \quad (4.13)$$

The choice $a_0 = 0, b_0 \neq 0$ leads to nonzero even coefficients for the b_i and for all $a_i = 0$. The reverse choice gives the only nonzero coefficient to be a_0 . Clearly both are just nonsense solutions.

From the point of view of a third order system the scalar-vacuum case does not pass the Painlevé test, does not possess the Painlevé Property and so is not integrable in the sense of Painlevé. However, we do note that there does exist a peculiar solution of the type mentioned by Ince [21, p 355] in the context of an equation discussed by Chazy in 1912. Naturally this solution would be acceptable were the system of the second order. This leads to the approach discussed in the following subsection.

4.1.2 The scalar-vacuum case as a second order system plus a first order system

An alternate approach to the treatment of systems (4.1) and (4.2) is to regard (4.1) as the two dimensional system of first order equations

$$\begin{aligned}\dot{x} &= -x^2 + \frac{1}{2}\lambda xy - 2y^2 + 1 \\ \dot{y} &= -3xy + \frac{1}{2}\lambda y^2 + \frac{1}{2}\lambda,\end{aligned}\tag{4.14}$$

where $x = \dot{a}$ and $y = \dot{\phi}$. This system can be treated independently of (4.2) since the latter contains the two constants of integration following from the integration of x and y . (We note that this approach was used by Halliwell [20] to treat (4.14) as a two-dimensional dynamical system.)

The usual substitution for the leading order behaviour of (4.14) gives the following patterns of exponents

$$\begin{array}{cc} p-1 & 2p \\ q-1 & p+q \end{array} \quad \begin{array}{cc} p+q & 2q \\ p+q & 2q. \end{array}\tag{4.15}$$

The values $p = q = -1$ are consistent for both sets. This is the only possible pattern since, although $p = -1, q > -1$ is fine for the first set, it fails for the second since the left side of (4.15a) becomes zero and so no balancing can occur and the reverse happens with $p > -1, q = -1$. The coefficients of the leading order terms satisfy

$$\begin{aligned}-\alpha &= -\alpha^2 + \frac{1}{2}\lambda\alpha\beta - 2\beta^2 \\ -\beta &= -3\alpha\beta + \frac{1}{2}\lambda\beta^2\end{aligned}\tag{4.16}$$

from the second of which it follows that

$$\alpha = \frac{1}{3} + \frac{1}{6}\lambda\beta\tag{4.17}$$

and from the first of which with (4.17)

$$(\lambda^2 - 36)\beta^2 + 4\lambda\beta + 4 = 0\tag{4.18}$$

We recall that $\lambda > 0$. Consequently the coefficients of the leading order terms are given by

$$\alpha = \frac{1}{6}\beta = -\frac{1}{6} \quad \text{if } \lambda = 6\tag{4.19}$$

$$\alpha = \mp\beta, \quad \beta = \frac{-2}{\lambda \pm 6} \quad \text{if } 0 < \lambda \neq 6.\tag{4.20}$$

This means that for general λ there are two possible solutions to be considered for consistency with the requirements of the Painlevé test.

The resonances are the eigenvalues of the equation

$$\begin{bmatrix} r-1+2\alpha-\frac{1}{2}\beta\lambda & -\frac{1}{2}\lambda\alpha+4\beta \\ 3\beta & r-1+3\alpha-\lambda\beta \end{bmatrix} \begin{bmatrix} \mu \\ \nu \end{bmatrix} = 0 \quad (4.21)$$

which follows from the substitution

$$x = \alpha\tau^{-1} + \mu\tau^{r-1} \quad \text{and} \quad y = \beta\tau^{-1} + \nu\tau^{r-1} \quad (4.22)$$

into the dominant terms of the system (4.14). We obtain the generic $r_1 = -1$ no matter the value of λ and $r_2 = \frac{2}{3}$ in the case that (4.19) applies and $r_2 = \pm 8/(\lambda \pm 6) = \mp 4\beta$ in the case that (4.20) applies. Since the nondominant terms in (4.14) are constants, the expansion must be in a power of τ such that it be commensurate with 1 and the second resonance, r_2 . This involves a constraint on the permissible values which λ may take. It is impossible for both values of r_2 to belong to the interval $(0, 1)$ and so in this range the system (4.14) cannot pass the Painlevé test. However, the requirements of commensurability and two positive values of r_2 can be satisfied for some $\lambda \in (0, 6)$ and the potential of both branches to pass the test is present. The possession of the Painlevé Property by (4.14) is not possible in the case $r_2 = 1$, i.e., $\lambda = 2$ since at that value the arbitrary constant that is introduced has zero coefficient and cannot be used to compensate for the presence of the nondominant constant term in (4.14). The constant terms deny the possibility of a Left Painlevé series and are responsible for the restriction on the interval of λ for which interval one can hope to obtain Right Painlevé series for both solutions.

Note that possession of the Painlevé Property by (4.14) means that the original second order pair (4.1) does not possess it since both a and ϕ will contain a logarithmic singularity due to the quadrature of the leading order terms. Nevertheless a solution of lessened analytic quality will exist and one must now consider the effect of the constraint (4.2). After a little manipulation the combination $x(4.14a) - y(4.14b)$ leads to

$$x\dot{x} - y\dot{y} = \frac{1}{2} (x^2 - y^2 - 1) (-2x + \lambda y) \quad (4.23)$$

which is immediately integrable to

$$x^2 - y^2 - 1 = cst \exp(-2a + \lambda\phi). \quad (4.24)$$

Consequently there is a further constraint on the coefficients of the expansions for $x(\tau)$ and $y(\tau)$ determined by the constant of integration in (4.24). This constant is the value of k , $= 0, \pm 1$, and this will fix the single arbitrary constant found in the series (apart from the location of the singularity).

For example, in the case that $\lambda = 10$ so that $r_2 = 1/2$, the expansions are

$$x(t) = \sum_{n=0}^{\infty} a_n \tau^{n/2-1}, \quad y(t) = \sum_{n=0}^{\infty} b_n \tau^{n/2-1}. \quad (4.25)$$

The result is surprisingly nice as the first few coefficients are

$$\begin{aligned}
a_0 &= 1/8, & b_0 &= -1/8 \\
a_1 &= 3b_1, & b_1 &= b_1 \\
a_2 &= 2b_1^2, & b_2 &= -2b_1^2 \\
a_3 &= -24b_1^3, & b_3 &= -8b_1^3 \\
a_4 &= \frac{1}{6} (11 - 192b_1^4), & b_4 &= \frac{1}{6} (13 + 192b_1^4)
\end{aligned} \tag{4.26}$$

and we obtain

$$\begin{aligned}
x(t) &= \frac{1}{8} (t - t_0)^{-1} + 3b_1 (t - t_0)^{-1/2} + 2b_1^2 - 24b_1^3 (t - t_0)^{1/2} + O(t - t_0) \\
y(t) &= -\frac{1}{8} (t - t_0)^{-1} + b_1 (t - t_0)^{-1/2} - 2b_1^2 - 8b_1^3 (t - t_0)^{1/2} + O(t - t_0).
\end{aligned} \tag{4.27}$$

Substitution in the last constraint equation seems to be a formidable task. The left hand side is

$$\begin{aligned}
LHS &= b_1 (t - t_0)^{-3/2} + 8b_1^2 (t - t_0)^{-1} + 8b_1^3 (t - t_0)^{-1/2} - 1 - 128b_1^4 \\
&\quad - 128b_1^5 (t - t_0)^{1/2} + 512b_1^6 (t - t_0) + O(t - t_0)^{3/2}
\end{aligned} \tag{4.28}$$

and the right hand side, up to the constant of integration,

$$\begin{aligned}
RHS &= (t - t_0)^{-3/2} + 8b_1 (t - t_0)^{-1} + 8b_1^2 (t - t_0)^{-1/2} - 128b_1^3 \\
&\quad - 480b_1^4 (t - t_0)^{1/2} + \frac{1792}{5}b_1^5 (t - t_0) + O(t - t_0)^{3/2}.
\end{aligned} \tag{4.29}$$

Consequently and therefore we can only conclude that, although there is agreement in the two series for the initial terms, this agreement fails for higher terms in the Laurent expansions. In the neighbourhood of the singularity the two would appear to be almost equal and in a numerical integration one could be tempted to ascribe the discrepancies to numerical error and to input integrability in the sense of Painlevé to a systems which is not integrable in that sense. (One recalls a similar suggestion of integrability in the surface of section plots for the Hénon-Heiles problem at low energies, the equivalent to being close to the singularity in the present instance.)

4.2 The scalar-fluid case

This is the case where a general (flat or curved) FRW model has a scalar field (with an exponential potential) and a separately conserved perfect fluid. The equations for the flat case are [22] [Eqns (6)-(7)]

$$x' = -\frac{3}{2}(2 - \gamma)x + \lambda\sqrt{\frac{3}{2}}y^2 - 3(2 - \gamma)x^3 - 3\gamma xy^2 \tag{4.30}$$

$$y' = -\frac{3}{2}y - \lambda\sqrt{\frac{3}{2}}xy - 3(2 - \gamma)x^2y - 3\gamma y^3 \tag{4.31}$$

which are derived from the system

$$\begin{aligned}\dot{h} &= -\frac{1}{2}\kappa^2 (\rho_\gamma + p_\gamma + \dot{\phi}^2) \\ \dot{p}_\gamma &= -3H (\rho_\gamma + p_\gamma) \\ \ddot{\phi} &= -3H\dot{\phi} - V'(\phi)\end{aligned}\tag{4.32}$$

subject to

$$p_\gamma = (\gamma - 1)\rho_\gamma\tag{4.33}$$

$$V = V_0 e^{-\lambda\kappa\phi}\tag{4.34}$$

$$H^2 = \frac{1}{3}\kappa^2 \left(\rho_\gamma + \frac{1}{2}\dot{\phi}^2 + V \right)\tag{4.35}$$

which represent the assumed equation of state, the assumed form of the potential and the constraint following from the Einstein Field Equations, by the definitions

$$x = \frac{\dot{\phi}\kappa}{\sqrt{6}H}, \quad y = \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \quad \tau = \ln a\tag{4.36}$$

and the prime denotes differentiation with respect to new time, τ . In terms of the solutions of the system (4.30) and (4.31) the constraint equation (4.35) now becomes the definition of the variable $\kappa\sqrt{\rho_\gamma}/\sqrt{3}H$ and so the treatment of the system (4.30) and (4.31) can be undertaken without the necessity to consider any constraint.

We substitute

$$x = \alpha\tau^p \quad y = \beta\tau^q\tag{4.37}$$

to obtain the set of exponents

$$\begin{array}{ccccc} p-1 & p & 2q & 3p & p+2q \\ q-1 & q & p+q & 2p+q & 3q \end{array}\tag{4.38}$$

from which it is evident that there are two possible patterns of leading order behavior, *viz* $p = q = -\frac{1}{2}$ involving the left side and final two terms of both equations and $p = 0, q = -\frac{1}{2}$ involving the second and fourth terms of the right side of (4.30) and the left side and final term of the right side of (4.31). The second pattern does not fit into the scheme of the Painlevé test and so the possession of the Painlevé Property depends only upon the first pattern. We note that the third order terms in both equations can also balance since they share the symmetry $x\partial_x + y\partial_y$ [12], but this leads to some constant solutions which are not comparable with the remaining terms in the two equations.

Case $p = q = -\frac{1}{2}$:

From the dominant terms of both equations the coefficients are required to satisfy the single equation

$$\gamma\beta^2 = (2 - \alpha)^2 - \frac{1}{3},\tag{4.39}$$

which is very encouraging for the passing of the Painlevé test. To determine the resonances we substitute

$$x = \alpha\tau^{-\frac{1}{2}} + \mu\tau^{r-\frac{1}{2}} \quad y = \beta\tau^{-\frac{1}{2}} + \nu\tau^{r-\frac{1}{2}} \quad (4.40)$$

into the dominant terms to obtain the two dimensional system

$$\begin{bmatrix} r+1-3\gamma\beta^2, & 3\alpha\beta\gamma \\ -3(2-\gamma), & r+3\gamma\beta^2 \end{bmatrix} \begin{bmatrix} \mu \\ \nu \end{bmatrix} = 0 \quad (4.41)$$

in which (4.39) has been used to make some simplification.

The system (4.41) has a nontrivial solution if $r = -1, 0$ which is consistent with (4.39) and means that the system (4.30) and (4.31) possesses the (weak) Painlevé Property since the nondominant terms do not enter until higher powers.

For the other pattern of behaviour we substitute

$$x = \sum_{i=0}^{\infty} \tau^{\frac{1}{2}i} \quad \text{and} \quad y = \sum_{i=0}^{\infty} \tau^{\frac{1}{2}i-\frac{1}{2}}. \quad (4.42)$$

We find that the odd coefficients are all identically zero and that the only arbitrary constant is the location of the movable singularity (in y), t_0 . Consequently we have a peculiar solution.

In terms of the definitions of the new variables in terms of the original variables the constraint equation, (4.35), gives $\kappa\sqrt{\rho_\gamma}/\sqrt{3}H$ as a function with a square root singularity at t_0 analytic, away from the single branch cut, in the variable τ . Unfortunately the definition of new time makes all functions nonanalytic in terms of the original time variable. However, there is a sense of integrability provided one keeps to the new time. It is not possible to make even this remark about the original dependent variables since it is the three combinations of them which have been shown to have the moderately good behaviour.

The equations for the case where a general curved FRW model has a scalar field with an exponential potential and a separately conserved perfect fluid are given by van den Hoogen *et al* [23] [Eqns (1.10-1.12)]. From the fourth order system

$$\begin{aligned} \dot{H} &= -\frac{1}{2} \left(\gamma\rho_\gamma + \dot{\phi}^2 \right) - K \\ \dot{\rho}_\gamma &= -3\gamma H\rho_\gamma \\ \ddot{\phi} &= -3H\dot{\phi} + \kappa V \end{aligned} \quad (4.43)$$

subject to the constraint

$$H^2 = \frac{1}{3} \left(\rho_\gamma + \frac{1}{2}\dot{\phi}^2 + V \right) + K \quad (4.44)$$

we obtain the third order system

$$x' = -2x + Ay^2 + Bx\Omega + 2x^3 - xy^2 \quad (4.45)$$

$$y' = y - Axy + By\Omega + 2x^2y - y^3 \quad (4.46)$$

$$\Omega' = -2B\Omega + 2B\Omega^2 + 4x^2\Omega - 2y^2\Omega, \quad (4.47)$$

in which we have written $A = \kappa\sqrt{\frac{3}{2}}$ and $B = \frac{3}{2}\gamma - 1$ to make the expressions look slightly simpler.

Under the usual substitution for the leading order behaviour we find the following pattern of exponents

$$\begin{array}{cccccc} p-1 & p & 2q & p+r & 3p & p+2q \\ q-1 & q & p+q & q+r & 2p+q & 3q \\ r-1 & r & & 2r & 2p+r & 2q+r. \end{array} \quad (4.48)$$

It is evident that the general singularity pattern is $p = q = -\frac{1}{2}, r = -1$. There are also other patterns of a more specialised nature. We list them in Table I.

p	q	r	Observation
$-\frac{1}{2}$	$> -\frac{1}{2}$	> -1	not P
$> -\frac{1}{2}$	$-\frac{1}{2}$	> -1	not P
$> -\frac{1}{2}$	$> -\frac{1}{2}$	-1	not P
$-\frac{1}{2}$	$-\frac{1}{2}$	> -1	P if $r = -\frac{1}{2}$
$-\frac{1}{2}$	$> -\frac{1}{2}$	-1	not P
$> -\frac{1}{2}$	$-\frac{1}{2}$	-1	not P.

Table I: Possible sundominant patterns of semisingular behaviour for the system 4.45 - 4.47.

Only the fourth case is a candidate for the Painlevé test as the other possibilities do not conform to the singularity requirement of the function and/or its derivatives.

In the general case the coefficients of the leading order terms satisfy the single constraint

$$4\alpha^2 - 2\beta^2 + 2B\gamma + 1 = 0 \quad (4.49)$$

so that one would expect to find the resonances $r = -1, 0(2)$. With the use of (4.49) the characteristic equation for the resonances is

$$\begin{vmatrix} r - 4\alpha^2 & 2\alpha\beta & -B\alpha \\ -4\alpha\beta & r + 2\beta^2 & -B\beta \\ -8\alpha\gamma & 4\beta\gamma & r - 2B\gamma \end{vmatrix} = 0$$

$$\Leftrightarrow r^3 + r^2 = 0 \quad (4.50)$$

so that $r = -1, 0(2)$ as anticipated from (4.49) so that there is consistency. There are three arbitrary constants and the nondominant terms can cause no theoretical difficulties. For this pattern of singular behaviour (4.45) - (4.47) passes the Painlevé test.

The other candidate for the Painlevé test, case four of Table I, with $p = q = r = -\frac{1}{2}$ produces the system

$$\begin{aligned} -\frac{1}{2}\alpha &= 2\alpha^3 - \alpha\beta^2 \\ -\frac{1}{2}\beta &= 2\alpha^2\beta - \beta^3 \\ -\frac{1}{2}\gamma &= 4\alpha^2\gamma - 2\beta^2\gamma \end{aligned} \tag{4.51}$$

for the coefficients of the leading order terms. There is a contradiction between the first two equations of (4.51) on the one hand and the third equation on the other. Consequently this is not an admissible case for the system (4.45) - (4.47). The one admissible singularity pattern has passed the Painlevé test and so we conclude that the system (4.45) - (4.47) possesses the (weak) Painlevé Property. The constraint equation, (4.44), for the original fourth order system. (4.43), defines the fourth variable. The same comments as for the system (4.32) apply to this case also.

5 The general flat case

We use the notation of Foster [24]. He presents the dynamical system

$$\begin{aligned} \frac{d\phi}{dt} &= \dot{\phi} \\ \frac{d\dot{\phi}}{dt} &= -K\dot{\phi} - V'(\phi) \\ \frac{dK}{dt} &= -\frac{3}{2}\dot{\phi}^2 \end{aligned} \tag{5.1}$$

subject to the constraint

$$K^2 = 3V(\phi) + \frac{3}{2}\dot{\phi}^2. \tag{5.2}$$

Foster introduces the new variables

$$x = \frac{1}{K}, \quad y = \sqrt{\frac{3}{2}} \frac{\dot{\phi}}{K} \quad \text{and} \quad \tau = \log v(t) + \tau_0 \tag{5.3}$$

where K is the trace of the extrinsic curvature and $v(t)$ represents volume, $v \sim a^3$. Then the system becomes

$$\begin{aligned}\frac{dx}{d\tau} &= y^2 x \\ \frac{dy}{d\tau} &= -y - \sqrt{\frac{3}{2}} x^2 V'(\phi) + y^3 \\ \frac{d\phi}{d\tau} &= \sqrt{\frac{2}{3}} y\end{aligned}\tag{5.4}$$

and the constraint is now

$$y^2 + 3x^2 V(\phi) = 1.\tag{5.5}$$

In order to make some reasonable progress with the Painlevé analysis of this system we require some explicit functional form for the potential, $V(\phi)$. (We note that in Foster's Dynamical Systems approach [24] such a constraint was not necessary. All that was required were some general properties of the potential. The approaches via Dynamical Systems and Singularity Analysis are complementary.) We make the standard *Ansatz* for a confining potential (*cf* the transition from (3.1) to (3.2)), *viz*

$$V(\phi) = \frac{1}{2} m^2 \phi^2\tag{5.6}$$

so that the system (5.5) is now

$$\begin{aligned}\frac{dx}{d\tau} &= y^2 x \\ \frac{dy}{d\tau} &= -y - \sqrt{\frac{3}{2}} x^2 m^2 \phi + y^3 \\ \frac{d\phi}{d\tau} &= \sqrt{\frac{2}{3}} y\end{aligned}\tag{5.7}$$

and the constraint is now

$$y^2 + \frac{3}{2} x^2 m^2 \phi^2 = 1.\tag{5.8}$$

We make the usual substitution for the leading order behaviour to obtain

$$\alpha p \tau^{p-1} = \alpha \beta^2 \tau^{p+2q}\tag{5.9}$$

$$\beta q \tau^{q-1} = -\beta \tau^q + \beta^3 \tau^{3q} - \sqrt{\frac{3}{2}} \alpha^2 \gamma^2 m^2 \tau^{2p+r}\tag{5.10}$$

$$\gamma r \tau^{r-1} = \sqrt{\frac{2}{3}} \beta \tau^q\tag{5.11}$$

From (5.9) we have: $q = -\frac{1}{2}$, α is arbitrary, p arbitrary and $\beta^2 = p$. With this (5.11) gives $r = \frac{1}{2}$ and $\gamma = 2\sqrt{2/3}\beta$. In (5.10) the possibility that $p < -1$

cannot occur as there would only be one term and so there can be no balance. If $p > -1$, the two dominant terms require that $\beta^2 = -\frac{1}{2}$ and from above this means $p = -\frac{1}{2}$, but α remains arbitrary. If $p = -1$, $\alpha^2 = -1/4m^2$.

The system (5.7) permits the two patterns of dominant behaviour, *viz* $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ and $(-1, -\frac{1}{2}, \frac{1}{2})$. For the first pattern of dominant behaviour the resonances are found to be $r = -1, -\frac{1}{2}, 0$. The first is the generic value. The third reflects the arbitrary value of α . The second demonstrates that this pattern cannot pass the Painlevé test since it requires a Left Painlevé Series and this is inconsistent with the nondominant terms in the original system (5.7) which require a Right Painlevé Series. The series expansion for this pattern of dominant behaviour contains only two arbitrary constants and consists of even terms only. There are two expansions depending upon whether a positive or negative root is taken. The first few terms of the series based on the positive root are given by

$$\begin{aligned} x &= a_0 \tau^{-\frac{1}{2}} + \frac{1}{2} a_0 (1 + 2m^2 a_0^2) \tau^{-\frac{1}{2}} \\ y &= \frac{i}{\sqrt{2}} \tau^{-\frac{1}{2}} - \frac{i\sqrt{2}}{4} (1 + 2m^2 a_0^2) \tau^{-\frac{1}{2}} \\ \phi &= \frac{2i}{\sqrt{3}} \tau^{\frac{1}{2}} - \frac{i\sqrt{3}}{9} (1 + 2m^2 a_0^2) \tau^{-\frac{3}{2}} \end{aligned} \quad (5.12)$$

For the second pattern of dominant behaviour we find that $r = -1(3)$ which means that there is only one arbitrary constant, the location of the moveable singularity. The series expansion consists of even terms only and for the positive root the first few terms are

$$\begin{aligned} x &= \frac{i}{2m} \tau^{-1} + \frac{3i}{16m} + \frac{155 - 243i}{9216m} \tau + \dots \\ y &= i\tau^{-\frac{1}{2}} - \frac{3i}{16} + \frac{155}{4608} \tau^{\frac{1}{2}} + \dots \\ \phi &= 2\sqrt{\frac{2}{3}} i\tau^{\frac{1}{2}} - \frac{i}{16} \sqrt{\frac{2}{3}} \tau^{\frac{3}{2}} + \frac{31}{3204} \sqrt{\frac{2}{3}} \tau^{\frac{5}{2}} \dots \end{aligned} \quad (5.13)$$

Neither pattern of dominant behaviour passes the Painlevé test and so the system (5.7) does not possess the Painlevé Property.

6 Scaling solutions

Billyard *et al* [25] discuss the stability of cosmological scaling solutions within the class of spatially homogeneous cosmological models with a perfect fluid subject to a linear equation of state and a scalar field with an exponential potential. This type of study was extended by van den Hoogen *et al* [23] to Robertson-Walker spacetimes with nonzero curvature. Miritzis *et al* [4] have already shown that the corresponding problems in flat space possess the

Painlevé Property. For the former problem Billyard *et al* reduced the governing equations to the three-dimensional system

$$\begin{aligned}\dot{x} &= \sqrt{\frac{3}{2}}\kappa y^2 + \frac{3}{2}x [(\gamma - 2)\Omega - 2y^2] \\ \dot{y} &= 3y - \sqrt{\frac{3}{2}}\kappa xy + \frac{3}{2}y [(\gamma - 2)\Omega - 2y^2] \\ \dot{\Omega} &= -3(\gamma - 2)\Omega + 3\Omega [(\gamma - 2)\Omega - 2y^2].\end{aligned}\tag{6.1}$$

System (6.1) has a superficial resemblance to a cubic system, but this removed by the nonhomeomorphic transformation $y^2 = z$ which reduces (6.1) to the quadratic system

$$\begin{aligned}\dot{x} &= \sqrt{\frac{3}{2}}\kappa z + \frac{3}{2}x [(\gamma - 2)\Omega - 2z] \\ \dot{z} &= 6z - \sqrt{6}\kappa xz + 3z [(\gamma - 2)\Omega - 2z] \\ \dot{\Omega} &= -3(\gamma - 2)\Omega + 3\Omega [(\gamma - 2)\Omega - 2z].\end{aligned}\tag{6.2}$$

Since the transformation between the two systems is not homeomorphic, the possession of the Painlevé Property by the one does not guarantee its possession by the other. However, possession of it by (6.2) will mean that (6.1) will possess the so-called weak Painlevé Property since the transformation replaces a polelike singularity in z with a branch point singularity in y .

The standard substitution

$$x = \alpha\tau^p, \quad z = \beta\tau^q, \quad \Omega = \Gamma\tau^r\tag{6.3}$$

into the derivatives and quadratic terms of (6.2) gives the set of exponents

$$\begin{array}{cccc} p-1 & q & p+r & p+q \\ q-1 & p+q & q+r & 2q \\ r-1 & r & 2r & q+r. \end{array}\tag{6.4}$$

The third of (6.4) gives $q = r = -1$ and the second $p = -1$. The coefficients of the leading order terms are found from the solution of the linear system

$$\begin{aligned}-\alpha &= \frac{3}{2}(\gamma - 2)\alpha\Gamma - 3\alpha\beta \\ -\beta &= \sqrt{6}\kappa\alpha\beta + 3(\gamma - 2)\beta\Gamma - 6\Gamma^2 \\ -\Gamma &= 3(\gamma - 2)\Gamma^2 - 6\beta\Gamma.\end{aligned}\tag{6.5}$$

For nontrivial coefficients α , β and Γ we see immediately that there is an inconsistency between the first and third of (6.5). This can be resolved by setting either $p = 0$ or $r = 0$. However, the latter choice leads to further inconsistencies and we are forced to the set $p = 0$, $q = r = -1$ and so the system (6.1) cannot possess the Painlevé Property.

One could substitute the partially singular expansion

$$x = \sum_{i=0} a_i \tau^i, \quad y = \sum_{i=0} b_i \tau^{(i-1)/2}, \quad \Omega = \sum_{i=0} c_i \tau^{i-1} \quad (6.6)$$

into (6.1). The results are somewhat disappointing. The coefficients in the expansions can be expressed in terms of the one arbitrary coefficient, b_0 . In particular

$$a_0 = \kappa b_0^2 \quad \text{and} \quad c_0 = \frac{-1 + b_0^2}{3(\gamma - 2)}. \quad (6.7)$$

The series for y consists of even terms only. The solution depends upon two arbitrary constants only, b_0 and t_0 , and so is a peculiar solution.

The system derived for a self-interacting scalar field with an exponential energy density evolving in a Robertson-Walker spacetime containing a separately conserved perfect fluid is [25, 23]

$$\begin{aligned} \dot{x} &= -3x + \sqrt{\frac{3}{2}} \kappa y^2 + \frac{1}{2} x [(3\gamma - 2)\Omega + 2(1 + 2x^2 - y^2)] \\ \dot{y} &= -\sqrt{\frac{3}{2}} \kappa x y + \frac{1}{2} y [(3\gamma - 2)\Omega + 2(1 + 2x^2 - y^2)] \\ \dot{\Omega} &= -3\gamma\Omega + \Omega [(3\gamma - 2)\Omega + 2(1 + 2x^2 - y^2)] \end{aligned} \quad (6.8)$$

in a standard notation. This system, with $-K$ in place of κ , has already been given an extensive treatment in [4] and we simply summarise the results for completeness. The leading order exponents are $p = q = \frac{1}{2}$ and $r = -1$. Under the transformation

$$u = 4x^2, \quad v = -2y^2 \quad \text{and} \quad w = (3\gamma - 2)\Omega \quad (6.9)$$

the dominant terms of (6.8) constitute the system

$$\begin{aligned} \dot{u} &= u(u + v + w) \\ \dot{v} &= v(u + v + w) \\ \dot{w} &= w(u + v + w) \end{aligned} \quad (6.10)$$

which has the leading order behaviour [4]

$$u = u_0 \tau^{-1}, \quad v = v_0 \tau^{-1}, \quad w = w_0 \tau^{-1}, \quad (6.11)$$

where the coefficients are related according to $u_0 + v_0 + w_0 + 1 = 0$. That the resonances are $r = -1, 0(2)$ confirms that two of the coefficients of the leading order terms are arbitrary and these with t_0 provide the three arbitrary constants needed for the solution.

Returning to the original system (6.8) we have that the system passes the Painlevé test for the weak property. (There is no need to be concerned about

incompatibilities at the resonances since the arbitrary constants already appear in the leading order terms. There are other patterns of quasisingular behaviour similar to those listed in the table in [4]. These do not fall within the ambit of the Painlevé test and do not affect the possession by (6.8) of the weak Painlevé Property and so integrability in terms of functions with only branch point singularities (in the case of x and y ; Ω is analytic away from its simple pole).

7 Comments and conclusions

The analysis of the systems of differential equations in the previous Sections enables us to make some more general comments about the Painlevé property and its possible applications. Firstly, we note that all the systems considered here come from reductions of the Einstein equations of general relativity by the imposition of certain symmetry and other plausible (from the physical point of view) assumptions. That is, they correspond or represent *relativistic* systems. One major characteristic of such systems of differential equations is their covariance, *ie* the independence of their properties from the coordinate frame used. However, it may be possible that the use of the Painlevé analysis for a system written in different coordinates leads to different behaviour. Thus we might conclude that the Painlevé test is not an appropriate one to use when dealing with relativistic equations. One may argue then that this test stands at a similar level as that of using numerical methods conducive to the determination of the so-called Liapunov exponents to decide on the issue of chaoticity of certain generic classes of relativistic cosmologies.

Such an opinion is formulated, as the astute reader will appreciate, somewhat tongue in cheek. The literature abounds with studies devoted to the determination of closed form solutions of the Einstein equations for various models. The variables of the equations are real and, not surprisingly, the solutions sought are real. A real solution need not be analytic and yet be quite useful for the explication of a specific physical model. Painlevé requires solutions which are analytic in the complex plane of the independent variable. This is a much stronger condition and, as we have remarked above, is superficially in conflict with the concept of covariance in the field equations and whatever is derived from them. The covariant formulation has no regard for the preservation of analyticity. However, the Painlevé Property is not preserved under a general transformation. Consequently what one should be seeking amongst the plethora of possible coordinate representations is that frame in which the Painlevé Property can be found, if it is ever to exist for a given system. This, of course, raises a new line of research which it is not appropriate to follow in this paper and that is to determine which of the possible myriads of coordinate representations of the field equations of a given model will be that set of coordinates for which the Painlevé Property holds. Given that the system is integrable in terms of analytic functions in one frame, the lesser demands of Cosmology can be met in a wider variety of coordinate systems.

Acknowledgments

PGLL thanks the Director of GEODYSYC, Dr S Cotsakis, and the Department of Mathematics of the University of the Aegean for the provision of facilities during the course of this work and acknowledges the continuing support of the National Research Foundation of South Africa and the University of Natal.

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